P-solutions of linear and nonlinear interval parametric systems and applications

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Outline

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• Methods for determining P-solutions
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1. Introduction

Consider the following parametric systems of decreasing generality and complexity:

- **nonlinear interval parametric (NLIP) system**

  \[ f(x, p) = 0 \]

  \[ f_i(x_1, \ldots x_n; p_1, \ldots p_m) = 0, \quad p_j \in p_j = [-1, 1], \quad i = 1, \ldots, n \] \hfill (1.10)

  \( x_j \) are the variables and \( p_j \) are the parameters

- **linear interval parametric (LIP) system**

  \[ f(x, p) = A(p)x - b(p) = 0 \quad , \quad p \in p \] \hfill (1.11)

- **nonlinear parametric dependences (NLPD) systems**
linear parametric dependence (LPD) systems, where

\[ a_{ij}(p) = \alpha_{ij} + \sum_{\mu=1}^{m} a_{ij\mu} p_\mu, \quad b_i(p) = \beta_i + \sum_{\mu=1}^{m} \beta_{i\mu} p_\mu \]  \hspace{1cm} (1.11b)

We need the concept of the solution set \( \Sigma \) of (1.10) or (1.11) defined as follows

\[ \Sigma = \{ x : f(x, p) = 0, \ p \in p \} \]  \hspace{1cm} (1.12)

Known “interval solutions” to (1.10) or (1.11): (i) interval hull (IH) solution \( x^* \): the smallest interval vector containing \( \Sigma \); (ii) outer interval (OI) solution \( x \): any interval vector enclosing \( x^* \), i.e. \( x^* \subseteq x \) and (iii) inner estimation of the hull (IEH) solution \( \xi \): an interval vector such that \( \xi \subseteq x^* \).
Recently, a new type of solution to LPD systems (1.11), (1.11b) has been introduced [Kolev 2014] which is of the following parametric form

\[ x(p) = Lp + a, \quad p \in \mathbf{p} \]  

(1.13)

where \( L \) is a real \( n \times m \) matrix whereas \( a \) is an interval vector. It is called a parameterized solution (\( p \)-solution) of (1.11). The \( p \)-solution is an outward linear approximation of \( \Sigma \).

Iterative methods for determining \( x(p) \) were suggested for the LPD case in [Kolev 2014, 2016 Rel. Comp.]; a simple direct method was proposed in [Kolev 2016 JACM]. Also the NLPD case was considered in [Skalna and Hladik 2017].
2. Properties of P-solutions

Let \( x(p) \) be a p-solution of system (1.10). Interval hull \( x(p) \) of \( x(p) \) - the smallest interval containing \( x(p) \).

- **Property P1.** The interval hull \( x(p) \) of \( x(p) \) is an outer interval solution \( x \) of (1.10), i.e. \( x = x(p) \).

Consider the \( i \)th component \( x_i^* \) of the IH solution \( x^* \) to (1.10) and the \( i \)th component \( x_i(p) \) of \( x(p) \)

\[
x_i(p) = \bar{x}_i + L_i p + \hat{x}_i [-1,1], \quad p \in p
\]

(\( \bar{x}_i, \hat{x}_i \) - centre and radius, \( L_i \)- \( i \)th row of \( L \)). Compute the ends

\[
x_i = \bar{x}_i - \sum_j |L_j| - \hat{x}_i, \quad x_i = \bar{x}_i + \sum_j |L_j| + \hat{x}_i
\]

(3.2)
Introduce the intervals:

\[ e_{i}^{(l)} = [x_{i}, x_{i} + 2\hat{x}_{i}], \quad e_{i}^{(u)} = [\bar{x}_{i}, \bar{x} - 2\hat{x}_{i}] \]  

(3.4)

- **Property P2.** The intervals (3.4) contain the endpoints of \( x_{i}^{*} \), i.e.

\[ x_{i}^{*} \in e_{i}^{(l)}, \quad \bar{x}_{i} \in e_{i}^{(u)}. \]  

(3.5)

Thus, the intervals (3.4) provide two-sided bounds on the ends of \( x_{i}^{*} \).

- **Property P3.** Introduce the intervals

\[ \xi_{i} = \begin{cases}  
[e_{i}^{(l)}, e_{i}^{(u)}], & \text{if } e_{i}^{(l)} < e_{i}^{(u)} \\
\text{empty interval, otherwise} & 
\end{cases} \]  

(3.6)

Then \( \xi_{i} \) determines the ith component of the IEH solution of (1.10).
3. Methods for determining a $p$-solution

All known methods refer to LIP systems and have polynomial time complexity. Here we only mention the

3.1. Direct method of [Kolev 2016] (Method $K$)
which requires roughly $N_k = n^4 m$ arithmetic operations

Comparison with method of [Skalna 2006] (Method $S_k$)
Methods $K$ and $S_k$ have the same complexity. However, unlike method $K$, method $S_k$ has only Property P1.

3.2. A new method

It is applicable to the general type of NLIP systems

$$ f(x, p) = 0, \quad y \in p $$

(3.32)
Computational scheme.

1). An outer interval solution $\mathbf{x}^0$ is computed (using some standard method).

2). The variables $\mathbf{x}^0$ and $y$ in (3.32) are then treated as independent, so (3.32) is replaced by

$$f(x, p) = 0, \quad x \in \mathbf{x}^0, \quad p \in p$$

(3.34)

3). The variables $x_i$ are written in normalized form

$$x_i = \tilde{x}_i + \hat{x}_i q_i, \quad q_i \in q_i = [-1, 1]$$

(3.35a)

Using (3.35a), (3.34) is modified to

$$f(q, p) = 0$$

(3.37a)
4). Using affine arithmetic (AA) the nonlinear system (3.37a) is enclosed by the LIP system.

\[ l(q, p) = \tilde{l} + A^q q + A^p p + \hat{l}[-1,1] = 0, \quad q \in q, \quad p \in p \]  \hspace{1cm} (3.37b)

where \( A^q \) and \( A^p \) are matrices of size \( n \times n \) and \( n \times m \)

5). Solve for

\[ q = \tilde{q} + Lp + \hat{q}[-1,1], \quad p \in p \]  \hspace{1cm} (3.38)

6). Using (3.35a) and \( q \) we compute

\[ x(p) = \tilde{x} + Lp + \hat{x}[-1,1], \quad p \in p \]  \hspace{1cm} (3.40)

7). Let \( x \) be the interval hull of (3.40)

If \( x \subseteq x^0 \) \hspace{1cm} (3.40a)

then \( x(p) \) obtained by (3.35a) to (3.40) is a parameterized solution of (3.32)
If \( x \) is narrower than \( x^0 \) and the reduction is larger than a threshold \( \varepsilon_r \), then \( x \) is renamed \( x^0 \) and a new iteration can be resumed from (3.35) until some stopping criterion is met.

**Numerical example with** \( n = 2 \) and \( m = 2 \)

The method was programmed in MATLAB environment using the toolbox IntLab.8 [Rump 1999]. The AA arithmetic was implemented by the *affari* toolbox [Rump and Kashiwagi 2015]. The system (3.34) is

\[
 f_1(x, y) = -y_1 x_1 + x_2^2 = 0, \quad (3.46a)
\]

\[
 f_2(x, y) = y_2 f_{21}(x_1) + x_2^2 = 0, \quad f_{21}(x_1) = x_1 (0.5 x_1 - 1) - 1, \quad (3.46b)
\]

\[
 y \in \mathbf{y}, \quad x \in \mathbf{x}^0 \quad (3.46c)
\]
\[ y_1 \in y_1 = [15.8, 16.2], \quad y_2 \in y_2 = [18, 19.6] \]
\[ x_1^0 = [1.2, 1.7], \quad x_2^0 = [-5.8, -4.7] \]

**Determining a p-solution**

The related system (3.37b) is

\[
\tilde{\lambda} = \begin{pmatrix} -4.5137 \\ -1.7110 \end{pmatrix}, \quad A^q = \begin{pmatrix} -4.000 & -5.775 \\ 2.115 & -5.775 \end{pmatrix}, \quad A^p = \begin{pmatrix} 0.2900 & 0 \\ 0 & 1.1065 \end{pmatrix}, \quad \hat{\lambda} = \begin{pmatrix} 0.20125 \\ 0.54750 \end{pmatrix}
\]

Hence

\[
x(p) = \tilde{x} + Lp + \hat{x}[-1,1], \quad p \in p
\]

(3.47)
\[
\begin{align*}
\tilde{x} &= \begin{pmatrix} 1.5645850 \\ -4.9947248 \end{pmatrix}, \quad L = \begin{pmatrix} -0.0118561 & 0.0523712 \\ -0.0095526 & -0.0689328 \end{pmatrix}, \quad \hat{x} = \begin{pmatrix} 0.0877044 \\ 0.1192227 \end{pmatrix}
\end{align*}
\] (3.47a)

Finally
\[
\begin{align*}
x &= \begin{pmatrix} [1.4768806, 1.6243233] \\ [-5.1139475, -4.8755021] \end{pmatrix}
\end{align*}
\] (3.48)

and it is seen that \( x \subseteq x^0 \)

Thus, the LIP form (3.47) determines the \( p \)-solution sought.

The subsequent iterations of method M1 yield better enclosures.

Convergence is reached in \( k = 4 \) iterations (relative reduction of \( \hat{x}^0_i \) to \( \hat{x}_i \) is less 1\%)
Property 2. Determining the two-sided bound

We illustrate with $e^l_2 = [e^l_2, e^l_2]$. For $k=4$

$$e^l_2 = [-5.1139, -4.8755]$$

The lower end $x^*_2$ of the IH solution $x^*_2$ is approximately

$$x^*_2 = -5.0884$$

and it is seen that indeed

$$x^*_2 \in e^l_2$$
4. A class of global optimization problems

Find the global minimum

\[ f_0^* = \min f_0(x, p) \quad (4.1a) \]

\[ f_i(x, p) = 0, \quad p \in \mathbf{p}, \quad i = 1, 2, \ldots, n \quad (4.1b) \]

The \( p \)-solution of (4.1b) – effective to solve (4.1).

Each individual problem is set up by specifying the functions \( f_0(x, p) \) and \( f(x, p) \). For instance:
Linear programming (LP) problem

\[ f_0(p) = c^T(p)x(p) \]

where the constraint is the LIP system \( A(p)x = b(p) \) [Kolev 2016]

Numerical example

\[ c^T = (1, 1, 1) \]

If \( c = \pm e_k \) (\( e_k \) is the k-th column of the identity matrix) the LP problem determines the lower/upper end of \( x_k^* \)

A method for solving such problems is available [Kolev and Skalna 2017] which is based on expressing

\[ f_0(p) = \sum_i x_i(p), \quad p \in p \]
Numerical evidence shows that it is superior to standard methods wrt:

- enclosure efficiency: tightness of the approximation of the solution
- computational cost;
- applicability radius [Kolev 2014, 2016]: largest radius of the box within which the respective method is still applicable

5. Applications

- *Worst-case tolerance analysis for electric circuits* [Kolev 2002]
  - *Direct current circuits*
  - *Alternating current circuits*
• *Power consumption analysis* [Kolev 2011, 2013]
• *Truss analysis* [Muhanna et. Al. 2004, 2005; Popova 2006]

Each of the above problems consists in determining or bounding a component $x_k^*$ of the corresponding hull solution $x^*$. The latter problem can be solved as corresponding simple interval linear programming ($c = \pm e_k$) using the approach of Section 4.

• *Eigenvalue range determination*

The standard method [Kolev 2010] can be improved using corresponding $p$-solutions
6. Conclusions

- The new methods employing p-solutions are better than their standard counterparts.
- NLIP problems can now be addressed using the new method of Section 3.2.
- Further numerical evidence would be welcome.
- It is hoped that the present survey will help promote the use of the new $p$-solution approach for solving various engineering problems involving uncertainties and risks.

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