Stochastic Methods in Damage Detection

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Research Goal

- Wave propagation in random media
- Representation of solution by Fourier integral operators
- Stochastic properties of medium modeled through the phase functions and amplitudes of the FIOs
- Application to damage detection

Plan of talk:

- A plane strain test model
- Fourier integral operator representation
- Parameter identification
- Stochastic FIOs for hypothesis testing
- Testing for damage

Industrial partner and supporting institutions:
A Plane Strain Test Model

**Elastic body:** Infinitely extended slab with square base, material parameters of aluminum.

**Excitation:** Constant in $z$-direction, only transversal components nonzero.

More precisely, a centered dynamic force $f_0(t)$ acting along the $z$-axis with frequency 1 MHz.

$$f(x, y, z, t) = \begin{bmatrix} 0 \\ \delta(x, y)f_0(t) \\ 0 \end{bmatrix}, \quad u(x, y, z, t) = \begin{bmatrix} u_1(x, y, t) \\ u_2(x, y, t) \\ 0 \end{bmatrix}$$

**Result:** a plane strain problem with displacements $u_1, u_2$. 
**Lamé Representation**

*Lamé representation by pressure and shear wave potentials:*

\[ u_1 = \partial_x \Phi + \partial_y \Psi, \quad u_2 = \partial_y \Phi - \partial_x \Psi \]

Reduction to two wave equations:

\[ \partial_t^2 \Phi(x, y, t) - c_p^2 \Delta \Phi(x, y, t) = \varphi(x, y, t), \]
\[ \partial_t^2 \Psi(x, y, t) - c_s^2 \Delta \Psi(x, y, t) = \psi(x, y, t), \]

The wave speeds are

\[ c_p = \sqrt{\frac{(1 - \nu) E}{\rho (1 + \nu)(1 - 2\nu)}}, \quad c_s = \sqrt{\frac{E}{2\rho (\nu + 1)}}. \]

The functions \( \varphi \) and \( \psi \) can be computed from the data by solving

\[ \partial_x \varphi + \partial_y \psi = 0, \quad \partial_y \varphi - \partial_x \psi = f. \]
The pressure potentials can be expressed in terms of FIOs:

\[
\Phi(x, y, t) = \frac{1}{8\pi^2} \int_0^t \left( \int_{\mathbb{R}^2} e^{ix\xi + iy\eta}e^{ic_p(t-s)}\sqrt{\xi^2+\eta^2} \hat{\varphi}_0(\xi, \eta)f_0(s)\,d(\xi, \eta) \\
+ \int_{\mathbb{R}^2} e^{ix\xi + iy\eta}e^{-ic_p(t-s)}\sqrt{\xi^2+\eta^2} \hat{\varphi}_0(\xi, \eta)f_0(s)\,d(\xi, \eta) \right) \,ds
\]

and similarly for the shear potential.

In the stochastic case, space-dependent wave speeds \( c_p, c_s \) will be introduced:

\[
\Phi(x, y, t) = \frac{1}{8\pi^2} \int_0^t \left( \int_{\mathbb{R}^2} e^{ix\xi + iy\eta}e^{ic_p(x,y)(t-s)}\sqrt{\xi^2+\eta^2} \hat{\varphi}_0(\xi, \eta)f_0(s)\,d(\xi, \eta) \\
+ \int_{\mathbb{R}^2} e^{ix\xi + iy\eta}e^{-ic_p(x,y)(t-s)}\sqrt{\xi^2+\eta^2} \hat{\varphi}_0(\xi, \eta)f_0(s)\,d(\xi, \eta) \right) \,ds
\]
**Parameter Identification (1)**

*Parameters of the undamaged material:*

- Modulus of elasticity (mean and variance): \( E_0 = E_{\text{mean}}, \sigma_E \)
- Poisson’s ratio (mean and variance): \( \nu_0 = \nu_{\text{mean}}, \sigma_\nu \)

*Background random fields*

\[
E(x, y) = E_{\text{mean}} + R_E(x, y),
\]

\[
\text{cov}(R_E(x_1, y_1), R_E(x_2, y_2)) = \sigma^2_E \exp \left( -\frac{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}}{L_E} \right)
\]

with correlation length \( L_E \); and similarly for \( \nu(x, y) \).

*Sensors at eight locations:*

They record the time dependent signal \( u_1(x_j, y_j, t), \)
\( u_2(x_j, y_j, t), j = 1 \ldots 8. \)
Nominal Data Values

Body:
Aluminum cuboid with base 10 cm × 10 cm, $\rho = 2.70$ g/cm².

Data:

$E_0 = E_{\text{mean}} = 70$ GPa, $\sigma_E = 3.5$ GPa,
$
u_0 = \nu_{\text{mean}} = 0.35$, $\sigma_\nu = 0.005$
$L_E = L_\nu = 3$ cm.

Excitation:

Centered dynamic force $f_0(t) = \sin 2\pi t$ acting along the z-axis
with frequency 1 MHz, $T_{\text{max}} = 7\mu$s.

The dimensions of the domain are larger than $c_p \cdot T_{\text{max}}$ – no interference
with boundary conditions during the time interval under consideration.
**Virtual test set-up.** Data of the undamaged structure are generated by a finite element model, fed with the given parameters and the background random field.

The parameters $c_p, c_s$ of the Fourier integral operator model are to be calibrated using only the sensor data.

**Calibration of nominal values $E_0 = E_{\text{mean}}, \nu_0 = \nu_{\text{mean}}$:**

Minimizing the $L^2$-distance between FE-data and the FIO-output at each sensor and jointly, leading to estimates

$$\hat{E}_{0,j}, \hat{\nu}_{0,j}, \hat{E}_0, \hat{\nu}_0, j = 1, \ldots, 8.$$ 

**Calibration of variances; Poisson’s ratio:**

$$\hat{\sigma}^2_{\nu} = \text{var}(\hat{\nu}_0) = \frac{1}{8N - 1} \sum_{j=1}^{8} \sum_{k=1}^{N} \left( \hat{\nu}_{0,j}^{(k)} - \hat{\nu}_0 \right)^2$$

with a data sample size of $N = 100$. 
Due to the failure of the corresponding estimate for $\sigma_E^2$, an empirical scaling curve was derived.

Observations:

(A) $\sigma_{\hat{E}}$ depends linearly on $\sigma_E$ at fixed $L_E$;

(B) At fixed $\sigma_E$, a functional dependence $\sigma_{\hat{E}} = \sigma_E f(L_E)$ can be fitted;

(C) The correlation length $L_{\hat{E}_0}$ can be estimated by fitting a Gaussian random field to the data at the eight sensors, and a functional dependence $L_E = g(L_{\hat{E}_0})$ can be fitted.
In this way, one can estimate the correlation length and the variance of Young’s modulus

$$\hat{L}_E \quad \text{and} \quad \hat{\sigma}_E = \frac{\sigma \hat{E}}{f(g(L\hat{E}_0))}.$$ 

In conclusion, the parameters of the random field

$$E(x, y) = E_{\text{mean}} + R_E(x, y)$$

describing the undamaged material have been estimated.

**Next goal.** Using the calibrated FIO-model and its output to detect parameter changes in the material.

This is achieved by comparing Monte Carlo samples of the FIO-output of an undamaged material at the eight sensors with measurements of the response of a damaged material.
Null hypothesis: The material is in an undamaged state. Then the wave potentials are given by

\[
\Phi(x, y, t) = \frac{1}{8\pi^2} \int_0^t \left( \int_{\mathbb{R}^2} e^{ix\xi + iy\eta} e^{ic_p(x,y)(t-s)} \sqrt{\xi^2 + \eta^2} \hat{\varphi}_0(\xi, \eta) f_0(s) \, d(\xi, \eta) \right. \\
+ \left. \int_{\mathbb{R}^2} e^{ix\xi + iy\eta} e^{-ic_p(x,y)(t-s)} \sqrt{\xi^2 + \eta^2} \hat{\varphi}_0(\xi, \eta) f_0(s) \, d(\xi, \eta) \right) \, ds
\]

and similarly for \( \Psi(x, y, t) \), with the stochastic wave speeds

\[
c_p(x, y) = \sqrt{\frac{(1 - \nu(x, y)) E(x, y)}{\rho (1 + \nu(x, y))(1 - 2\nu(x, y))}}, \quad c_s = \sqrt{\frac{E(x, y)}{2\rho(\nu(x, y) + 1)}}.
\]

The random fields \( E(x, y) \) and \( \nu(x, y) \) are known (or estimated) from the data of the undamaged material.

By Monte Carlo simulation, a sample of \( N = 1000 \) responses of the undamaged material is generated.
Alternative hypotheses: The material is damaged. This is tested by comparing measured features with the 99%-confidence interval given by the sample.

Features: total spectral energy, phase angle and spectral energy at the first two DFT-frequencies in \(x\)- and \(y\)-direction at each sensor. Total number of features: 60.

1. Scenario 1: The material is having the desired properties (undamaged state).
2. Scenario 2: The material is not having the desired properties: \(E_{\text{mean}}\) is too small.
3. Scenario 3: The material is not having the desired properties: \(E_{\text{mean}}\) is too large.
4. Scenario 4: The material is having the desired properties, but is suffering a crack, modeled by a very small Young’s modulus.
(1) Scenario 1: The null hypothesis is not rejected. In most trials, all of the 60 features accept the null hypothesis.

(2) Scenario 2: The null hypothesis is rejected in almost all sensor locations (phase angle out of 99%-interval).

(3) Scenario 3: The null hypothesis is rejected in almost all sensor locations (phase angle out of 99%-interval).

(4) Scenario 4: The crack lies between sensor 3 and the origin. The sensor in location 3 diagnoses the damage (too large spectral energy, in total and at both frequencies), while the others do not.
Example of “good” signal:
Example of “bad” signal:

Signal at sensor 3, frequency 2 in x-direction

Phase angle

Spectral energy of freq

Total spectral energy

Data

Signal

Bounds
The program has been encouraged by earlier work of M. de Hoop, G. Papanicolaou, K. Sølna et al. on detecting Green’s function from signal cross-correlations.

The 3D homogeneous, isotropic case has been done as well.

In 1D and 2D, the computational cost is low enough to execute the program on standard computing equipment.

In 3D, high computational power is needed.

Theoretical research topics remaining:
(1) Reduction of computational cost.
(2) Choice of random fields.
(3) Fully non-isotropic and non-homogeneous case.