Epistemic vs Aleatory: Granular Computing and Ideas Beyond That

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1. What We Want: Two Types of Objectives

- Most practical problems come from the following two main objectives:
  - we want to *understand* the world, to learn more about it, and
  - we also want to *change* the world.
- Often, we pursue both objectives. For example:
  - we want to predict the path of a tropical storm,
  - and we want to come up with measures that will decrease the negative effects of this storm;
  - we need to decide which areas to evacuate, etc.

2. In Both Cases, We Get Systems of Equations

- To describe the state of the world means to describe the numerical values of the corresponding quantities.
- E.g., to describe a mechanical system, we need to describe the coordinates and velocities of all its objects.
- The values of some of these quantities can be easily measured.
- However, we cannot directly measure future values of the quantities.
- We need to predict them based on the known dependence between the current and future values.
- In some practical cases, we have explicit formulas that enable us to make this prediction.

3. We Get Systems of Equations (cont-d)

- However, in most other cases, we do not have such explicit formulas.
- Instead, we have a system of equations that relates current and future values of the corresponding quantities.
- Sometimes, these equations include the values of related auxiliary quantities.
- Example: Newton's theory does not have explicit formulas for
  - predicting the position of a new comet in the next month
  - based on its position at the present moment.

4. We Get Systems of Equations (cont-d)

- Instead, it has equations that:
  - describe the position of a comet at any given moment of time
  - as a function of to-be-determined parameters of the corresponding orbit.
- We equate the observed locations of the orbit with the results predicted by this formula.
- Thus, we get a system of equations from which we can find these parameters.
- Then, we can then use a similar equation to predict the future location of the comet.

5. We Get Systems of Equations (cont-d)

- In general:
  - let us denote the measured quantities by $x = (x_1, \ldots, x_n)$,
  - let us denote the desired quantities by $y = (y_1, \ldots, y_m)$,
  - let us denote the auxiliary quantities by $z = (z_1, \ldots, z_p)$.
- In these terms, the corresponding system of q equations has the form $F_i(x, y, z) = 0$, $i = 1, \ldots, q$.
- In this system of equations:
  - we know x,
  - the values y and z are unknowns that need to be determined from the above system, and
  - we are only interested in the values y.

6. Systems of Equations That Come From the Desire to Change the World

- In such problems, the goal is to achieve a certain desired state of the world by making appropriate changes.
- E.g., determining how to correct the trajectory of a spaceship so that it reaches the destination.
- E.g., finding the parameters of an engine that satisfy the desired efficiency and pollution levels.
- In general:
  - let $x = (x_1, \ldots, x_n)$ denote the parameters describing the current state of the world,
  - let $t = (t_1, \ldots, t_s)$ denote the parameters that described the desired state, and
  - let $y = (y_1, \ldots, y_m)$ denote the values of the parameters that describe the sought-for intervention.

7. Changing the World (cont-d)

- In some cases, we have an explicit formula that determines the future state of the world:

$$G(x, y) = (G_1(x, y), \ldots, G_s(x, y)).$$

- In such cases, to find the proper intervention, we must solve the system of equations

$$G_i(x, y) = t_i, \quad i = 1, \ldots, s.$$

- In this system of equations:
  - we know x and t,
  - the values y are the unknowns that need to be determined from the above system, and
  - we are interested in the values y.

8. Changing the World (cont-d)

- In other cases, we only have an implicit relation between x, y, and the future state:

$$F_i(x, t, y, z) = 0, \quad i = 1, \ldots, q.$$

- Here $z = (z_1, \ldots, z_p)$ are auxiliary quantities.
- In this system of equations:
  - we know x and t,
  - the values y and z are unknowns that need to be determined from the system, and
  - we are only interested in the values y.

9. Need to Take Granularity into Account

- In the above description, we implicitly assumed that all the known values are known exactly, both:
  - values x that come from measurements or
  - values t that describe what we want.
- In reality, in both cases, instead of the exact value, we have a *granule*.
- For example, measurements are never absolutely accurate, there is always some measurement uncertainty.
- For describing what we want, we also need granules.
- For example, when we set a thermostat on 25° C, it does not mean that we want exactly 25.0.
- We will not notice small differences.
- A more adequate representation of our objective is an interval like $[24, 26]$.

10. Simplest Granule: a Set

- In some cases, based on the measurement result:
  - we know exactly which actual values are possible and which are not possible, and
  - we also know exactly which states we want and which we do not want.
- In such cases, the corresponding information about x (and/or t) consists of describing:
  - which values are possible (correspondingly, desirable) and
  - which are not.
- In other words, the proper description of the corresponding granularity is a *set*:
  - a set X of possible current states of the world, and
  - a set T of all desired states.

11. What If We Have No Information About Some States

- Sometimes, after a measurement:
  - for some states x, we know that they are possible,
  - for some states x, we know that they are not possible, and
  - for some states x, we have no idea whether they are possible or not.
- This situation can be naturally described by a "set interval" $[\underline{X}, \overline{X}]$, where:
  - $\underline{X}$ is the set of all states x about which we know that they *are* possible, and
  - $\overline{X}$ is the set of all the states x about which we know that they *may be* possible,
  - i.e., about which we do not know that they are not possible.

12. Set Intervals (cont-d)

- Similarly, when we describe our desires:
  - for some states t, we know that they are desirable,
  - for some states t, we know that they are not desirable, and
  - for some states t, we have no idea whether they will be desirable or not.
- This situation can be naturally described by a set interval $[\underline{T}, \overline{T}]$, where:
  - $\underline{T}$ is the set of all states t about which we know that they *are* desirable, and
  - $\overline{T}$ is the set of all the states t about which we know that they *may be* desirable,
  - i.e., about which we do not know that they are not desirable.

13. Need to Take Into Account Degrees of Possibility

- Often, for some states for which we are not 100% sure that these states are possible:
  - an expert can come up with a degree – e.g., a number from the interval $[0, 1]$
  - indicating to what extend this particular state x is possible.
- This additional information is a function that assigns, to each state, a degree. It is called a *fuzzy set.*
- For different states t about which we are not sure whether they are desirable or not:
  - we can often come up with a degree
  - to which each such state is desirable.
- In this case, the set of all desirable states also becomes a fuzzy set.

14. Probabilistic Uncertainty

- For possible states, we can also use prior experience of similar situations.
- Thus, we come up with frequencies with which different states x occurred.
- So, we have a *probability distribution* on the set of all possible states – i.e., a probabilistic granule.

15. More Complex Granules Are Also Possible

- In addition to the above basic types of granules, we can also have more complex granules.
- We can have type-2 fuzzy sets, in which the degree of possibility or desirability:
  - is not a real number
  - but is itself a fuzzy subset of the interval $[0, 1]$.
- We can have p-boxes, in which:
  - instead of a single probability distribution,
  - we have a family of probability distributions, etc.

16. Resulting Problem

- We have mentioned that many practical problems can be reduced to solving systems of equations, in which:
  - we know the values x (and t), and
  - we need to find the values y.
- In practice, instead of the exact values of x (and t), we now have granules X (and T).
- So, we need to decide how to solve the systems of equations in such a granular case.

17. What We Show in This Talk

- At first glance, the situation is straightforward: all we need to do is to find out:
  - how to extend the usual solution algorithms
  - to the corresponding interval, fuzzy, etc., case.
- There are indeed known techniques for extending algorithms to the interval, fuzzy, etc. cases.
- In many cases, these extensions work well, but in many other cases, they don't.
- In this talk, we explain why they don't:
  - it is not enough to consider the corresponding mathematical equations,
  - we need to know where these equations came from.

18. What We Show in This Talk (cont-d)

- This need will be illustrated mainly on the example of interval uncertainty – the simplest type of uncertainty.
- The main intent of this talk is pedagogical: to help users avoid common mistakes.
- We give a simple example of two different practical problems in which:
  - the equations are the same,
  - the granules are the same, but
  - the practical relevant solutions are different.

19. Since Our Intent Is Pedagogical, We Select the Simplest Possible Examples

- Our intent, as we have mentioned, is to help the user deal with uncertainty – and avoid possible mistakes.
- From this viewpoint, we are trying to illustrate our point on the simplest possible examples, in which both:
  - the uncertainty is of the simplest possible type – namely, interval uncertainty, and
  - the equations are the simplest possible: in both example, we take $a = b + c$.

20. First Practical Problem

- We have the amount a of water in a reservoir.
- We then release the amount b.
- We would like to know the amount of water c left in the reservoir.
- The solution to this simple problem is straightforward:

$$c = a - b.$$

- For example, for $a = 100$ and $b = 40$, we have

$$c = 100 - 40 = 60.$$

21. Second Practical Problem

- We have the amount a of water in the reservoir, which is too large.
- We want to release some amount c so that, as a result, we will only have the amount b left.
- How much water should we release?
- The solution to this second simple problem is also straightforward: $c = a - b$.
- This is exactly the same formula as for the first practical problem.
- For example, for $a = 100$ and $b = 40$, we get the same solution $c = 100 - 40 = 60$ as for the first problem.

22. Simple Granules

- For both above problems, we implicitly assumed that we know the exact values a and b.
- Let us now consider a more realistic situation, in which:
  - instead of the exact values a and b,
  - we have intervals A and B.
- As our first example, let us take

$$A = [99, 101] \text{ and } B = [38, 42].$$

- Let us see what happens in both problems.

23. First Practical Problem

- In the first problem:
  - all we know about the original amount of water a is that a is somewhere between 99 and 101, and
  - all we know about the released amount is that it was somewhere between 38 and 42.
- We want to find the range of possible values of the resulting amount $c = a - b$, i.e., the set

$$C = \{c = a - b : a \in [99, 101], b \in [38, 42]\}.$$

- The function $c = a - b$ is increasing in a and decreasing in b.
- Thus, the largest possible value of c is attained when a is the largest ($a = 101$) and b is the smallest ($b = 38$).
- The resulting largest possible value of c is thus equal to $c = 101 - 38 = 63$.

24. First Practical Problem (cont-d)

- The smallest possible value of c is attained when a is the smallest ($a = 99$) and b is the largest ($b = 42$).
- The resulting largest possible value of c is thus equal to $c = 99 - 42 = 57$.
- Thus, the desired interval of possible values of c is equal to $C = [57, 63]$.

25. Second Practical Problem

- In the second problem:
  - all we know about the original amount of water a is that a is between 99 and 101, and
  - we want to make sure that after releasing the amount c, the remaining amount is between 38 and 42.
- In other words, we need to find the values c for which:
  - no matter what was the original value $a \in [99, 101]$,
  - the remaining amount $b = a - c$ will be between 38 and 42.
- Let us describe the set of all such values c.
- We want the value c for which the double inequality $38 \le a - c \le 42$ holds for all $a \in [99, 101]$.
- By reversing signs, we get an equivalent double inequality $-42 \le c - a \le -38$.

26. Second Practical Problem (cont-d)

- By adding a to all three sides of this inequality, we get an equivalent inequality $a - 42 \le c \le a - 38$.
- The left inequality means that c should be larger than or equal to $a - 42$ for all $a \in [99, 101]$.
- This is equivalent to requiring that c is larger than or equal to the largest of these differences.
- The difference is the largest when a is the largest, i.e., when $a = 101$.
- Thus, the left inequality is equivalent to

$$c \ge 101 - 42 = 59.$$

- The right inequality means that c should be smaller than or equal to $a - 38$ for all $a \in [99, 101]$.

27. Second Practical Problem (cont-d)

- This is equivalent to requiring that c is smaller than or equal to the smallest of these differences.
- The difference is the smallest when a is the smallest, i.e., when $a = 99$.
- Thus, the right inequality is equivalent to

$$c \leq 99 - 38 = 61.$$

- So, in this problem, the desired interval of possible values of c is equal to $C = [59, 61]$.

28. These Solutions Are Different

- The interval $[57, 63]$ corresponding to the first problem is much wider than $[59, 61]$.
- This is not a mistake.
- For example, the value $c = 63$ is a possible solution of the first problem.
- It corresponds to the case when we originally had $a = 101$, and we released $b = 38$.
- However, the same value $c = 63$ is *not* a possible solution to the second problem:
  - indeed, if we had $a = 99$,
  - then by releasing $c = 63$ units of water, we would be left with $b = a - c = 99 - 63 = 36$ units,
  - and we want the remaining amount to be always between 38 and 42.

29. Second Example

- A simple modification can make the different between the first and second problems even more drastic.
- To get such a modification, let us take use different interval granules: $A = [98, 102]$ and $B = [39, 41]$.
- The difference is not so big for the first problem.
- In this case, we want to find the range of possible values of the resulting amount $c = a - b$, i.e., the set

$$C = \{c = a - b : a \in [98, 102], b \in [39, 41]\}.$$

- The function $c = a - b$ is increasing in a and decreasing in b.
- So, the largest possible value of c is attained when a is the largest ($a = 102$) and b is the smallest ($b = 39$).
- The resulting largest possible value of c is thus equal to $c = 102 - 39 = 63$.

30. Second Example (cont-d)

- The smallest possible value of c is attained when a is the smallest ($a = 98$) and b is the largest ($b = 41$).
- The resulting largest possible value of c is thus equal to $c = 98 - 41 = 57$.
- Thus, the desired interval of possible values of c is equal to $C = [57, 63]$.
- In the second practical problem:
  - all we know about the original amount of water a is that a is somewhere between 98 and 102, and
  - we want to make sure that after releasing the amount c, the remaining amount is between 39 and 41.

31. Second Example (cont-d)

- Thus, we need to find the values c for which:
  - no matter what was the original value $a \in [98, 102]$,
  - the remaining amount $b = a - c$ will be between 39 and 41: $39 \le a - c \le 41$.
- By reversing signs, we get an equivalent inequality $-41 \le c - a \le -39$, i.e., $a - 41 \le c \le a - 39$.
- For $a = 98$, the right side of this double inequality implies that $c \le 98 - 39 = 59$, so $c \le 59$.
- On the other hand, for $a = 102$, the left side of this inequality implies that $c \ge 102 - 41 = 61$, so $c \ge 61$.
- But a number cannot be at the same time larger than or equal to 61 and smaller than or equal to 59.
- Thus, for $A = [98, 102]$ and $B = [39, 41]$, the second practical problem simply has no solutions to all.

32. Lesson Learned

- There are many papers that:
  - first, come up with algorithms for solving, e.g., systems of linear equations under uncertainty, and
  - then, apply these algorithms to all the cases.
- We hope that the above two examples convinced the readers that it is not possible to just know the equation.
- We need to take into account what exactly practical problem is being solved.
- Depending on that, different solutions will be adequate.

33. So, What Do We Do?

- In general, instead of knowing the exact state x (or t), we only know the set X (or T) of possible states.
- For the understanding the world, a natural idea is to find all possible values y, i.e., to find the set $Y =$

$$\{y : \exists x \in X \, \exists z \in Z \, (F_1(x, y, z) = 0 \,\&\, \ldots \,\&\, F_q(x, y, z) = 0)\}.$$

- This set combines ("unites") all the values y corresponding to all possible values $x \in X$.
- It is thus known as the *united solution set.*
- For changing the world, we need to find the values y for which:
  - for all possible values $x \in X$,
  - the resulting vector t is within the desired range T.

34. So, What Do We Do (cont-d)

- In this case, the desired set Y has the form

$$Y = \{y : \forall x \in X\, \exists t \in T\, \exists z \in Z\, (F_1(x,t,y,z) = 0\, \&\, \ldots)\};$$

  - when we select the control parameters values y from this set Y,
  - the resulting state t is guaranteed to belong to the set T of desirable (tolerable) sets.

- Because of this, this solution is known as the *tolerance solution set*.

35. How to Actually Find Solutions

- In general, the corresponding problems are NP-hard, even under interval uncertainty.
- However, in many cases, there are efficient algorithms for solving these problems.
- The most well-studied problem is the problem of finding the united solution set.
- The simplest algorithm for solving this problem is the *naive interval computation* algorithms, in which:
  - we start with an algorithm for solving the system, and
  - we replace each elementary arithmetic operation with the corresponding interval operation.

36. How to Find Solutions (cont-d)

- These operations can be easily determined via monotonicity, like we described the range of $a - b$:
  - if we know that the value a belongs to $[\underline{a}, \overline{a}]$ and that the value b belongs to $[\underline{b}, \overline{b}]$,
  - then the set $[\underline{c}, \overline{c}]$ of possible values of the difference $c = a - b$ can be computed as

$$[\underline{c}, \overline{c}] = [\underline{a} - \overline{b}, \overline{a} - \underline{b}].$$

- This fact can be described as

$$[\underline{a}, \overline{a}] - [\underline{b}, \overline{b}] = [\underline{a} - \overline{b}, \overline{a} - \underline{b}].$$

37. How to Find Solutions (cont-d)

- Similarly, for other arithmetic operations, the corresponding ranges can be described as follows:

$$[\underline{a},\overline{a}]+[\underline{b},\overline{b}]=[\underline{a}+\underline{b},\overline{a}+\overline{b}];$$

$$[\underline{a},\overline{a}]\cdot[\underline{b},\overline{b}]=[\min(\underline{a}\cdot\underline{b},\underline{a}\cdot\overline{b},\overline{a}\cdot\underline{b},\overline{a}\cdot\overline{b}),\max(\underline{a}\cdot\underline{b},\underline{a}\cdot\overline{b},\overline{a}\cdot\underline{b},\overline{a}\cdot\overline{b})];$$

$$[\underline{a},\overline{a}]/[\underline{b},\overline{b}]=[\underline{a},\overline{a}]\cdot(1/[\underline{b},\overline{b}]),\ \text{where}$$

$$1/[\underline{b},\overline{b}]=[1/\overline{b},1/\underline{b}]\ \text{when}\ 0\notin[\underline{b},\overline{b}].$$

- The resulting enclosure is often a drastic overestimation.
- So more efficient methods need to be used: central value method, monotonicity checking, bisection, etc.
- Methods of computing tolerance solutions are sometimes called *modal interval mathematics*.

38. How to Find Solutions (cont-d)

- The reason for this name is that:
  - the main difference from the traditional interval computations (that computes the united solution)
  - is that one of the existential quantifiers is replaced by the universal one.
- This is similar to the usual interpretation of modalities like "possible" and "necessary", in which:
  - "possible" is understood as occurring in one of the possible worlds (which corresponds to $\exists$), while
  - "necessary" is understood as occurring in all possible worlds (which corresponds to $\forall$).
- Intervals $[\underline{t}_i, \overline{t}_i]$ corresponding to inverse modality can be formally viewed as *improper* (*Kaucher*) intervals

$$[\overline{t}_i, \underline{t}_i] \text{ with } \overline{t}_i > \underline{t}_i.$$

39. How to Find Solutions (cont-d)

- Kaucher intervals are indeed useful in solving the corresponding tolerance problem.
- For example, in the second problem:
  - we know that $a \in [\underline{a}, \overline{a}]$,
  - we are given the tolerance interval $[\underline{b}, \overline{b}]$, and
  - we want to find the value c for which $b = a - c \in [\underline{b}, \overline{b}]$ for all $a \in [\underline{a}, \overline{a}]$.
- Arguments like the ones that we had lead to the following interval of possible value of c:

$$[\underline{c}, \overline{c}] = [\overline{a} - \overline{b}, \underline{a} - \underline{b}].$$

- This is exactly what we get if we apply naive formula to the improper interval $A^* \stackrel{\text{def}}{=} [\overline{a}, \underline{a}]$ and to $[\underline{b}, \overline{b}]$.
- Similar ideas can be used to solve more complex systems of equations.

40. How These Methods Help to Solve Our Two Practical Problems: First Example

- The only information that we have about the quantity a is that it is in the interval $A = [99, 101]$.
- The only information that we have about the quantity b is that it is in the interval $B = [38, 42]$.
- In the first practical problem, we need to find the range C of all possible values $c = a - b$ when $a \in A$ and $b \in B$.
- In other words, we need to find the set

$$C = \{c : \exists a \in A \, \exists b \in B \, (c = a - b)\}.$$

- This is a particular case of the united solution set.
- As we have mentioned, to compute this set, we can use naive (straightforward) interval computations.
- Specifically, the computation of c consists of a single arithmetic operation (subtraction).

41. First Example (cont-d)

- According to the naive interval computation method, to compute the set C:
  - we replace this operation with numbers by the corresponding operation with intervals,
  - i.e., we compute

$$C = A - B = [99, 101] - [38, 42] = [57, 63].$$

- This is exactly the range that we obtained earlier.
- In the second practical problem, we need to find the range C of all possible values c for which:
  - for all $a \in A$,
  - the value $b = a - c$ belongs to the interval B.
- In other words, we need to find the set

$$C = \{c : \forall a \in A\, \exists b \in B\, (c = a - b)\}.$$

42. First Example (cont-d)

- This is a particular case of the tolerance solution.
- As we have mentioned, to compute this set, we can use Kaucher arithmetic.
- The variable a enters this formula with a universal quantifier instead of the existential one.
- So, instead of the original interval $A = [99, 101]$, we need to consider an improper interval $A^* = [101, 99]$.
- For the resulting pair of intervals A^* and B, the above general rule of interval subtraction leads to

$$C = A^* - B = [101, 99] - [38, 42] = [59, 61].$$

- This is exactly the range we found.

43. Second Example

- In the second example, we have

$$A = [98, 102] \text{ and } B = [39, 41].$$

- To solve the first problem, we perform naive interval computations and compute

$$C = A - B = [98, 102] - [39, 41] = [57, 63].$$

- This is exactly what we obtained.
- To compute the range corresponding to the second problem, we use Kaucher arithmetic and compute:

$$C = A^* - B = [102, 98] - [39, 41] = [61, 59].$$

44. Second Example (cont-d)

- What is the meaning of this answer?
- We are looking for all possible values c for which

$$\underline{c} \leq c \leq \overline{c}.$$

- In this example, $\underline{c} > \overline{c}$, so no such c are possible.
- Thus, for these intervals A and B, the second problem has no solutions.
- This is exactly the conclusion that we obtained earlier.

45. Other Formulations Are Possible

- in some practical problems, the control parameters y can be divide into two groups:
  - parameters y' that we select once and for all (e.g., the parameters that describe the design), and
  - parameters y'' that we can change all the time.
- In this case:
  - instead of a single desired state t,
  - it makes sense to consider different desired states that form a set T,
  - so that at different moments of time, we can reach different states.
- This is exactly the case with heating and air conditioning.

46. Other Formulations Are Possible (cont-d)

- It is usually set up in such a way that different users can set up different desired temperatures.
- In this case, when look for the original setting y', we must set it in such a way that:
  - any state from T
  - is accessible via an appropriate selection of y''.
- The resulting solution set has the following form $Y' =$

$$\{y' : \exists y'' \in Y'' \, \forall x \in X \, \exists z \in Z \, \exists t \in T \, (F_1(x, t, y, z) = 0 \, \& \, \ldots\}.$$

- This solution set is known as the *controlled set*.
- More complex sets appear in game-type situations, when participants make selections in turn.

47. Conclusion

- We showed that when a practical problem reduces to a system of equations, to find its solution:
  - it is not enough to know the corresponding granules,
  - we also need to take into account the original practical problem.